The Golden Phrase:
Steps to the Physics of Language[1]

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1. Introduction

This chapter is about some previously unnoticed (and, we suspect, significant) properties of the celebrated X-bar schema (Chomsky 1970, Jackendoff 1977). We observe that the X-bar schema is a uniquely special recursive format; this Golden Phrase is "just right" with respect to certain fundamental mathematical properties, of the kind that crop up repeatedly in the study of physical systems.[2,3] We focus on three properties that seem likely to be of significance, noting that the X-bar form is the only kind of structure with all three properties.

In familiar terms, the X-bar schema describes a recipe for constructing phrases (XPs) by combining their heads (X^0) with first one other phrase (YP, called the complement), then combining the resulting object (X') with a second phrase (ZP, called the specifier).

1)

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          XP
         /  \
        ZP   X'
            /  \
          X⁰    YP
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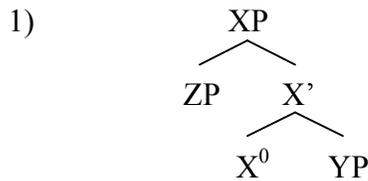
> "Arguably, this configurational schema, known as X-bar theory, is the only kind of structure that syntactic representations exploit. Other structural options, such as adjuncts to phrases, multiple specifiers of a single head, etc., have been experimented with in various ways but Cartographic research has, for the most part, eschewed these options, retaining only the core structures afforded by the X-bar schema. [...] The core structural relations defined by X-bar theory seem to be not only necessary, but sufficient to characterize syntactic structure."
> (Shlonsky 2010: 2)

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[1] The work reported here has been directly inspired by previous publications by Juan Uriagereka on Fibonacci patterns (see e.g. Uriagereka 1996, 1998; Idsardi and Uriagereka 2009; Piattelli-Palmarini and Uriagereka 2008) and by many of his bold ideas exchanged, over the years, in conversation.

[2] It is suggestive that the Golden Phrase – the X-bar schema – is intimately associated with the Fibonacci numbers and the Golden Mean, which appear in many other places in nature. Juan Uriagereka has been discussing Fibonacci patterns in linguistic structure for many years (see Footnote 1).

[3] For a close parallelism between the historical development and the conceptual foundations of linguistic theory and modern physics see Freidin and Vergnaud (2001), a paper that we find quite congenial.

However, recent developments within syntactic theory have undermined the theoretical foundation of X-bar theory, especially the notions of "bar-level" and projection on which it is based. This makes the empirical phenomenon all the more mysterious: within the vast morphospace of conceivable structural patterns, why should human grammars keep to this tiny corner?[4] In the context of the Minimalist Program (Chomsky 1995), we suspect that this structural pattern is not elaborately and specifically encoded as such, but rather emerges from the interaction of deeper principles.[5,6]

In this chapter, we show that the structural format of the X-bar schema, when compared against all other patterns that could provide a self-similar basis for binary-branching discrete infinity, stands out as special in a number of ways. The properties that set it apart are best brought to light by describing syntactic patterns with matrices; the matrix properties we discuss are familiar in other sciences, though to our knowledge their consideration in theoretical linguistics is novel (some might say, premature). Matrices seem, anyway, a natural choice for describing grammars, as they encode transformations and self-similar growth, two themes that loom large in recent work.[7]

In quantum physics it is standard to treat the initial state of a system as a vector and the application of forces, fields and perturbations as matrices. The result of this operation (a vector-matrix multiplication) is a new vector (the final state). Well-behaved (so to speak) vector transformations correspond to diagonalizable matrices (see below). The special vectors that preserve their orientation under the transformation expressed by matrix multiplication are called eigenvectors, and the factors by which these stable vectors grow (or shrink) in each step are called eigenvalues. These are computed by finding the roots of the characteristic polynomial of the matrix.

The special classifications that converge uniquely in the X-bar schema involve the spectrum of eigenvalues associated with the matrix representation A of a phrasal recurrence pattern. Let us call the degree of a phrasal pattern the degree of its characteristic polynomial. Let G (mnemonic for "growth factor") represent the dominant eigenvalue (i.e., the largest-magnitude, necessarily real, root of the characteristic polynomial), and G' stand for an arbitrary Galois conjugate (a distinct eigenvalue; equivalently, a distinct root of the characteristic polynomial). Among "Prime" systems

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[4] Obviously, phrasing the question this way presupposes that the X-bar schema is in fact an accurate description of syntactic structure. This is a fraught topic, depending in part on theory-internal concerns, and touching on controversial topics such as the nature of endocentricity (see e.g. Chomsky 2013, 2015 in press). Such matters are beyond the scope of this chapter.

[5] Of course, the enterprise of explaining X-bar-like syntactic formatting through deeper principles has a rich history. Notable here is Kayne's (1994) theory of Antisymmetry, in which a version of X-bar theory was made to fall out from the requirements of linearizing an unordered branching tree. For a different approach to antisymmetry in language, see Moro (2000, 2013).

[6] Chomsky has suggested to us, in recent conversations, that the remarkable properties of the X-bar structure which we detail here, rather than being part of Narrow Syntax, are part of the NS interface with the Conceptual-Intentional system, something about which very little is known. In fact, he now conceives the X-bar structure as emergent from recursive Merge. He was kind enough to say that the properties revealed in our present work are "more interesting than you think," precisely because of this.

[7] For the derivation of the X-bar schema and the Fibonacci series by repeated application of the binary Pauli matrices, see (Piattelli-Palmarini and Vitiello 2015a,b).

(those whose matrix forms have irreducible characteristic polynomials), the three classes of interest are as follows:

(i) *The Endocentric class*. These forms can be described as generalized X-bar schemata; intuitively, each combines a terminal at the deepest level with a complement phrase, and then combines the result with some number of specifier phrases one at a time, to make a full phrase. These forms have the largest G for systems of their degree; there is one such system of each degree (up to permutation of which non-terminal is chosen as the root). Relative to other patterns of their degree, these structural formats support the minimum number of c-command relations (Medeiros 2008).

(ii) *The Pisot class*. These patterns have a G that is a so-called Pisot-Vijayaraghavan (PV) number, an algebraic integer (i.e. solution to a polynomial with integer coefficients) greater than 1, with Galois conjugates (here, G') all of magnitude less than 1. Although discovered only in the twentieth century, these numbers have been the focus of considerable interest in number theory, harmonic analysis, and crystallography (see, e.g., Moody 1997). These patterns have a kind of structural purity; all eigenvectors (interpreted as a stable structural 'theme') save the dominant one vanish as the pattern is grown. All non-Pisot systems, in turn, have infinite growing "warts" of structure distinct from the dominant theme.

(iii) *The Polygonal class*. Finally, there is a class of patterns whose G is of the form $2\cos(\pi/n)$. These forms are polygonal: their G is the ratio of the shortest internal diagonal to a side in a regular polygon. They, and only they, have all real-valued G', and diagonalizable matrices. The odd polygonal systems (whose G relates to the geometry of a polygon with an odd number of sides) furthermore have symmetric matrices ($a_{ij} = a_{ji}$). In polygonal systems the growth of the pattern reflects real scaling of each of its components, thus an inhomogeneous dilation. On the other hand, all non-polygonal systems have eigenvectors (stable configurations) associated with some complex-valued G', and their growth involves a kind of rotation.

Figure 1 below is a Venn diagram of these classes. The intersection of all three classes is exhausted by a unique form, corresponding to the X-bar schema. We speculate that the unique convergence of these properties in this structural format might be (part of) the reason "why" language is that way. If so, X-bar-like phrase structure might follow from the "third factor" in language design (Chomsky 2005), the result of very general principles (namely, those related to the mathematics of matrices, a natural way of capturing syntactic patterning, and arising in too many physical applications to mention).[8]

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[8] For a similar, germane, though distinct, mathematical and physical approach to "third factors," see Piattelli-Palmarini and Vitiello (2015a,b).
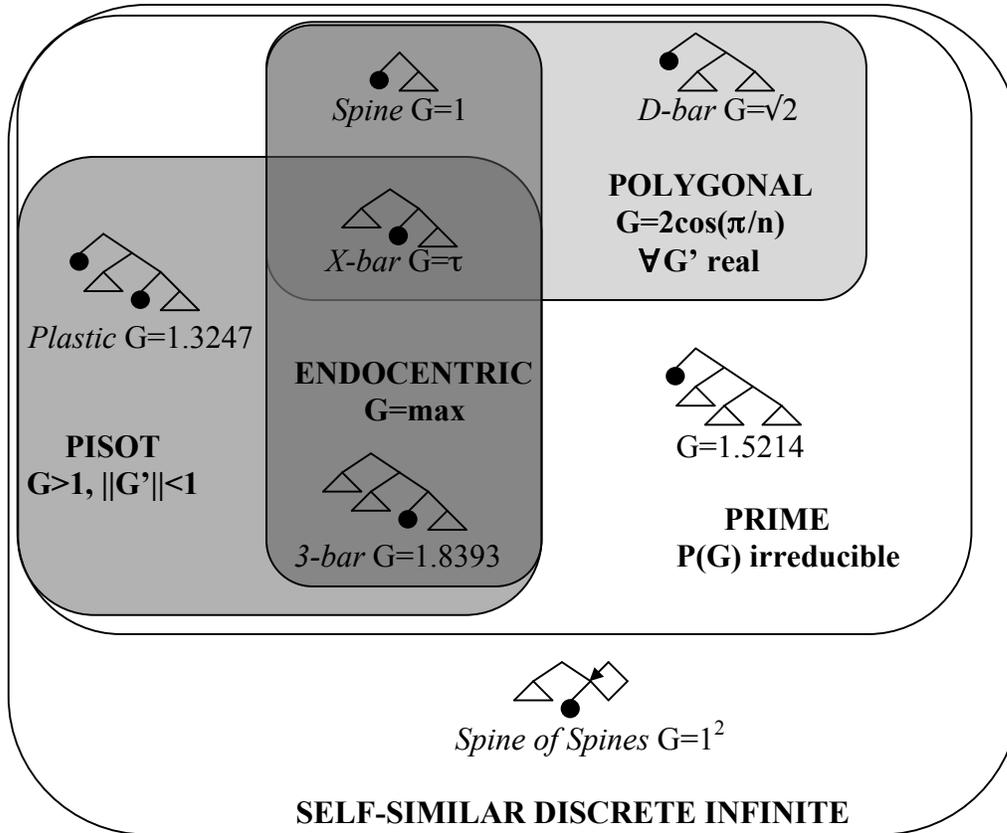
Figure 1. Intersection of spectral classes of phrasal recurrence patterns.

The X-bar schema, whose dominant eigenvalue G is the so-called Golden Number τ, is the only pattern in all three of the classes described here (Endocentric, with maximal G for its degree; Pisot, with unique structural purity; and Polygonal, with real growth).

2. Expressing syntactic patterns as matrices

The X-bar schema is just one of (infinitely) many patterns that could form a structural basis for language-like structures, combining terminals and non-terminals into indefinitely large structures. We consider the X-bar schema against the background of other binary-branching deterministic context-free patterns of syntactic recurrence. As a way of coming to grips with the consequences of different ways of building structure, we will isolate the patterns and consider what happens as they are expanded rigidly, deterministically, and unboundedly.

Formally, we take a pattern of syntactic recurrence to be a tuple consisting of an alphabet containing a single terminal and some finite number n of non-terminals (without loss of generality, we may label the terminal with 0, and non-terminals with distinct natural numbers i, $n \geq i \geq 1$), a designated root non-terminal (again without loss of generality, we can designate the root with the largest-value non-terminal symbol n), and a set of binary production rules i → j k (a one-to-one function associating to each non-

terminal a distinct unordered pair of elements drawn from the alphabet). We will further impose the conditions that each element of the alphabet is dominated by the root, and each non-terminal dominates the terminal type.

Thus the class of objects under consideration here form a restricted class of D0L-grammars (deterministic context-free Lindenmayer systems), used in many fields, including algorithmic botany, dynamical systems, and the study of quasi-crystals.[9] In this case we will consider them as idealizations of possible patterns of syntactic recurrence, each a potential scaffolding for the expressions of natural language.

Each phrasal pattern can be expressed as a matrix A recording immediate domination relations among non-terminals, a_{ij} the number of objects of type j immediately dominated by the object of type i. Thus, the 1st, 2nd, 3rd row/column is associated with the first/second/third non-terminal type; the rows are in effect the inputs to rewrite rules, and the columns the outputs. Thus, the X-bar form corresponds to the following matrix:

2) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Just to make the correspondence between the matrix form and the phrase structure rules absolutely clear, we can think of the matrix as a rewrite table, as follows:

3)

	XP	X'
XP →	1	1
X' →	1	0

As mentioned above, we designate with G the dominant eigenvalue of this matrix; it is the limiting ratio of the number of nodes on successive lines of the tree, as the pattern is maximally expanded. In turn, G' are non-dominant eigenvalues (Galois conjugates, other roots of the characteristic polynomial of the matrix). In syntactic terms, the other eigenvectors are other stable configurations among non-terminals.

We can distinguish "prime" phrasal patterns from "composite" ones. Prime patterns with n non-terminals have irreducible characteristic polynomials of degree n (and, so, their G, the largest magnitude root of the polynomial, is an algebraic integer of degree n). Composite patterns have polynomials that can be factored into polynomials of lesser degree; these systems can be described as geometrically substituting one pattern within another. It follows that the G of a composite pattern over n non-terminal types is an algebraic integer of degree $< n$.[10]

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[9] There is a large literature on L(indemayer)-systems; see e.g. Prusinkiewicz et al (1990) for an introduction. Much of that work is concerned with word sequences and formal language properties, and is thus irrelevant to the concerns here: we ignore linear order, and with only a single terminal type, all systems concerned here produce the dull (terminal) formal language a^n. Although matrix formulations of L-systems are well-established (see for example Rozenberg and Salomaa 1980), we are not aware of prior work that has investigated the properties considered here.

[10] The simplest example of a composite pattern is the "Spine of Spines", defined over two non-terminals. It can be expressed as a traditional rewrite system XP → XP X', X' → X^0 X'); its polynomial is $x^2 - 2x - 1$, which factors as $(x - 1)^2$, reflecting its geometric description as Spines substituted in at the head positions of a Spine (the Spine has polynomial $x - 1$).

To illustrate the utility of matrix mathematics for describing the properties of syntactic patterns, consider the maximal expansion of the X-bar pattern, as in Figure 2 below.

XP	X'
1	0
1	1
2	1
3	2
5	3
Fib(n)	Fib(n-1)
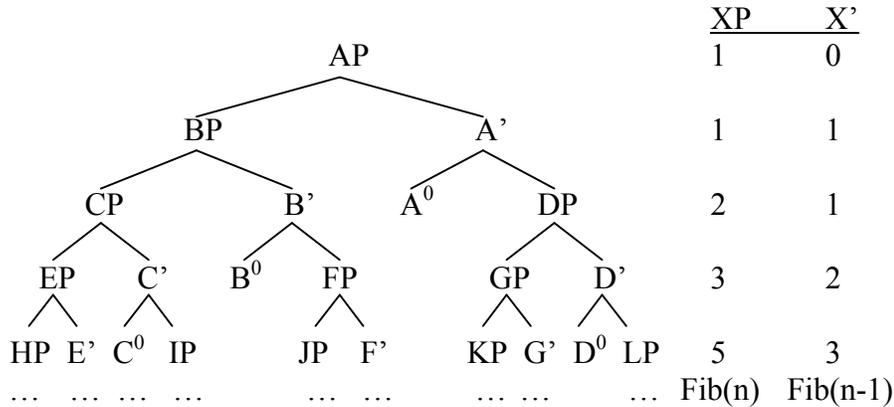
Figure 2. Maximal expansion of X-bar schema.

Each stage of line-by-line growth can be represented as a vector in the plane, associating the non-terminal types to the coordinate axes (the distance along each axis simply counts the number of that type of non-terminal on a line of the tree). To get the sequence of vectors $(x_0, x_1, ...)$ representing maximal expansion of the pattern, we simply iterate multiplication of the vectors by the phrasal matrix A:

4) $Ax_i = x_{i+1}$

Figure 3 below illustrates the first few steps of growth of the X-bar pattern.
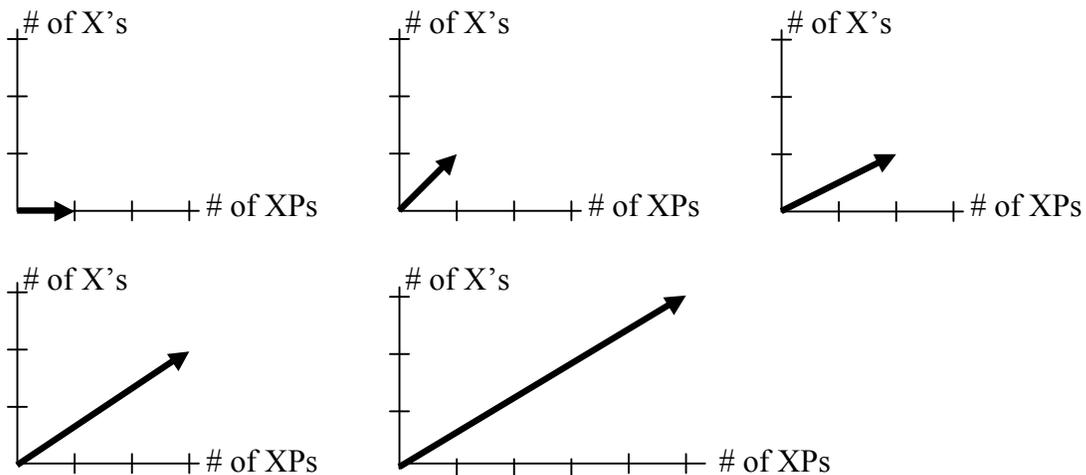
Figure 3: Vector representation of X-bar growth.

Below is the eigenvector for the X-bar scheme (the diagonal of a golden rectangle, as it turns out), with dominant eigenvalue $G = 1.618...$, the golden mean. Notice that the vectors representing subsequent levels in the tree converge on this direction.
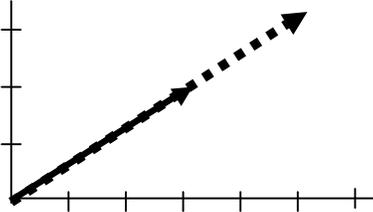
Figure 4: Eigenvector of the X-bar schema (solid line), and one step of growth (dashed line), corresponding to multiplication of the eigenvector by the X-bar matrix.

An eigenvector is a column vector *x*, such that multiplication by the matrix *A* simply scales the eigenvector by the associated eigenvalue λ. We illustrate this for the X-bar pattern and its dominant eigenvector x and associated eigenvalue λ in (5):

5) $Ax = \lambda x$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1.618... \\ 1 \end{bmatrix} = \begin{bmatrix} 2.618... \\ 1.618... \end{bmatrix} = 1.618... \begin{bmatrix} 1.618... \\ 1 \end{bmatrix}$

With this much in hand, we turn to the properties of each of the three special spectral classes; the first such is the Endocentric class.

3. The Endocentric class

The Endocentric class contains generalized X-bar forms, with one head and one complement at the deepest level of the phrase, and some fixed number of specifiers merged one-at-a-time above that, with complements and specifiers identical to the root object (in intuitive terms, all non-head daughters are phrasal). Such forms have the highest growth factor *G* for their degree (see Medeiros 2008 for an informal proof). For example, the X-bar schema grows faster than any other form with two non-terminals; 3-bar, the generalized X-bar form with two specifiers per phrase, has the largest *G* among all systems with 3 kinds of non-terminals. We give below one orientation of the 3-bar system, its matrix form, a description in terms of PSRs,[11] and an abstract tree diagram encoding the recurrence pattern. It has $G = 1.839...$, the so-called Tribonacci constant.[12]

3-bar

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad \begin{array}{l} XP \rightarrow XP\ X'' \\ X'' \rightarrow XP\ X' \\ X' \rightarrow X^0\ XP \end{array}$
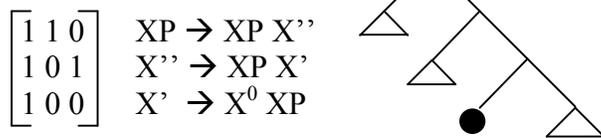
Figure 5. Matrix, rewrite rules, and tree schema for 3-bar, the Tribonacci grammar.

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[11] Note that this format corresponds to Jackendoff's (1977) "Uniform Three-Level Hypothesis", where all X-bar phrases were taken to project three layers of structure.

[12] The Tribonacci constant is the limiting ratio among successive terms of the Tribonacci sequence, where each term is the sum of the three preceding terms. Compare this to the case of the golden number τ, which is the limiting ratio of terms in the Fibonacci sequence, where each term is the sum of the previous two terms.

The Spine, though it is in fact the "worst" system – it has the lowest possible growth factor (1) for a binary-branching discrete infinite pattern – is also best in its class, by virtue of being the only kind of discrete infinite structure with one non-terminal. The alternatives to the Spine (1 → 1 0), namely 1 → 0 0 and 1 → 1 1 (where 1 is the non-terminal object and 0 the terminal), don't work: The first case (the Pair) is discrete but not infinite; the second is infinite but not discrete (the Bush, with no terminals).

The characteristic polynomial of an Endocentric system is one of the generalized Fibonacci polynomials, with this general form:

6) $x^n - x^{n-1} - ... - x - 1$

The X-bar system is, interestingly, the last endocentric system whose growth value can be physically constructed with straightedge and compass (Cipu and Luca 2001:28). See also Medeiros (2012) for the observation that, while Endocentric systems support the minimum number of c-command relations globally (over the whole tree), for Endocentric systems of degree 3 and beyond there are other patterns that accumulate fewer new c-command relations locally (within the horizon of a single phrase). In other words, the X-bar schema is the "last" Endocentric system that minimizes c-command relations both locally and globally.

4. The Pisot class

Next we turn to the Pisot class, containing patterns that have growth factors G that are Pisot-Vijayaraghavan numbers. These are numbers that are algebraic integers – solutions of polynomials with integer coefficients – where all but one of the roots lie within the unit circle on the complex plane, and the Perron-Frobenius eigenvalue is > 1.[13]

These systems are likely to be significant, for the following reason. Recall that we have identified the eigenvectors of these systems with stable ratios among non-terminal nodes. In Pisot systems, those eigenvectors with eigenvalues of magnitude less than one (i.e., all but the dominant one) are vanishing: their associated eigenvalue is less than one, so they shrink as the tree grows. Put another way, non-Pisot systems have growing "warts" of stable non-terminal configurations other than the dominant one, while in Pisot sytems all but the dominant ratio are configurations that extinguish themselves during growth.

The generalized X-bar form of degree ≥ 2 (i.e., all members of the Endocentric class save the Spine) is always of Pisot type. As mentioned above, its polynomial is the generalized Fibonacci polynomial, the largest root of which is always a Pisot number (Cipu & Luca 2001:27).

5. The Polygonal class

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[13] All integers greater than 1 are therefore Pisot numbers, trivially: integer n is the lone root of the degree 1 polynomial $x - n$. In what follows, we will be interested in non-integer Pisot numbers < 2. The golden mean τ is the lowest-degree non-integer Pisot number (it is the larger root 1.618... of $x^2 - x - 1$).

Finally we turn to the last of the three spectral classes to be described, the Polygonal class. In these systems, the growth factor G expresses the ratio of the shortest diagonal to a side in a regular n-gon. The ratio of shortest diagonal to side for a regular n-gon is this value:

7) $2*\cos(\pi/n)$

To see why, note that the exterior angle θ_1 is $2\pi/n$, because the exterior angles for the whole n-gon sum to a complete circle. The angle θ_2 in the diagram below is then π/n, half of the exterior angle.

Figure 6. Geometry of diagonal-to-side ratio in a regular polygon.

The cosine of angle $\theta_2 = \cos(\pi/n)$ then expresses the ratio of half a diagonal ($d/2$) to a side (s; see the triangle at right above); we double this to get the diagonal-to-side ratio stated above, $2*\cos(\pi/n)$.

The first, trivial case is an equilateral triangle; we may take the diagonals to coincide with the sides, so that the ratio of diagonal to side is simply 1. This growth factor is associated with the Spine. The Spine makes use of a single non-terminal type. With two non-terminal types, we get the D-bar and X-bar systems, with growth factors $\sqrt{2}$ and τ, respectively. These values, as it turns out, are the diagonal-to-side ratios for the next polygons, the square and pentagon.

In fact, we find the shortest-diagonal-to-side ratio for each regular n-gon as the growth factor of some phrase structure system. For example, the hexagon has diagonal-to-side ratio of $\sqrt{3}$; this is the growth factor for one family of systems with three non-terminal types. We also find the heptagon diagonal-to-side ratio among the class with three non-terminals; this is the value 1.8019... This pattern continues, with two polygonal systems of each degree (for degree n, these are the ($2n$)-gon and ($2n$+1)-gon).

Only these "polygonal" systems have diagonalizable matrices. Matrix A is diagonalizable if it is similar to a diagonal matrix D; this is so if there is an invertible matrix P such that the following relation holds:

8) $A = PDP^{-1}$

In contrast, all non-polygonal systems have non-diagonalizable matrix expressions; such matrices are called defective. This fact has a syntactic interpretation: in polygonal systems, the eigenvectors, corresponding to the stable configurations (ratios among the various kinds of non-terminals), all have real growth. The system as a whole thus represents an inhomogeneous dilation. Non-polygonal systems have some "complex" growth, and thus have an element of rotation in the growth of certain eigenvectors (stable syntactic configurations).

The systems corresponding to odd polygons appear to be even more special, in that their matrices (in all orientations, i.e. for any choice of which non-terminal to place at the

root of the tree) are symmetric. A symmetric matrix with real entries is a special case of the more general property of Hermitian matrices, which are their own conjugate transpose (where for arbitrary matrix element $a_{ij} = a + bi$, $a_{ji} = a - bi$ where i is the imaginary unit). In the search for a possible "physics of language", this sounds interesting, in light of the fact that the matrix operators representing observables in quantum theory (e.g., momentum, position, energy, etc.) are always Hermitian.

Diagonalizability and symmetry are important properties in a number of physical applications. Curiously, "polygonal" ratios in the sense described here have been found in the quasiperiodic oscillations of multiperiod variable stars. For example, Escudero (2003: 235-236) gives three examples of multivariable stars, giving values for the ratio of their two periods and polynomials that are, exactly, those of the pentagon, octagon, and nonagon (for stars UW Her(culis), ST Cam, and V Boo, respectively).

Figure 7 below summarizes the properties of polygonal phrasal patterns. Each row contains four related objects. In the leftmost column is a *G* value; the tree diagram in the rightmost column represents the syntactic pattern with this growth factor. The second column shows the regular *n*-gon where the indicated number is the ratio of shortest internal diagonal to a side. The third column is a matrix encoding the recurrence relations among non-terminals in the relevant syntactic pattern.

Number	Diagonal/side ratio in *n*-gon	Phrasal matrix	Syntactic pattern with this growth factor:

$2\cos(\pi/3)$
= 1

[1]

Triangle

Spine

$2\cos(\pi/4)$
= √2
1.4142…

$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

Square

D-bar

$2\cos(\pi/5)$
Φ = (1+√5/2)
1.6180…

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Pentagon

X-bar

$2\cos(\pi/6)$
= √3
1.7320…

$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Hexagon

$2\cos(\pi/7)$
1.8019…

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Heptagon

$2\cos(\pi/8)$
1.8478…

$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Octagon

$2\cos(\pi/9)$
1.8794…
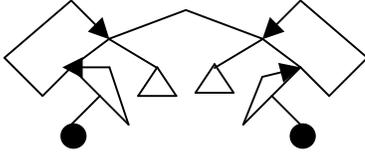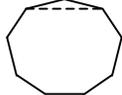
$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Nonagon
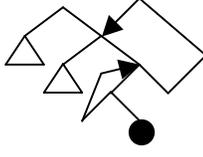
$2\cos(\pi/n)$
∈ [1, 2)

n-gon

$\begin{bmatrix} \text{Matrix} & 0 \\ \text{for } n\text{-}2 & … \\ \text{-gon} & 0 \\ & 1 \\ 0 … & 0\ 1\ 0 \end{bmatrix}$

$2\cos(\pi/\infty)$
= 2
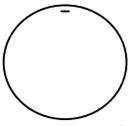
∞-gon (circle)

[2]
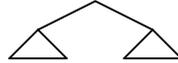
Bush

Figure 7. Summary of Polygonal properties.

6. Conclusion

In this chapter, we have described three special classes of syntactic patterns, defined in terms of the spectrum of eigenvalues of the matrix expressing each pattern. These are the Endocentric class, with largest growth factor (dominant eigenvalue) G of its degree; the Pisot class, where all non-dominant eigenvalues G' are of absolute value less than 1; and the Polygonal class, with all real eigenvalues and diagonalizable matrix forms (the odd polygonals furthermore have symmetric matrices). Each of these classes manifests properties that may be of significance for syntactic patterning.

Endocentric systems, as argued in Medeiros (2008), are desirable in that they support the minimum number of c-command relations. This means that such structures minimize the search space for the establishment of long-distance dependencies, a plausible desideratum for computational optimization (see Medeiros 2012 for details). Systems in the Pisot class are special in that they have only a single growing structural theme, with all non-dominant stable configurations vanishing under growth. Non-Pisot systems have infinite growing "warts" of stable configurations other than the dominant one. Finally, Polygonal systems have all real growth, encoding inhomogeneous dilations, by contrast with the complex rotations found in the growth of all other systems.

As we have stressed, these special spectral classes have a single pattern in common, the X-bar schema that seems to characterize the syntax of human language. This "Golden Phrase", which grows according to the golden mean, is thus a uniquely special structural format. We suggest that its unique cluster of spectral properties might be part of why it is found in syntactic structures, whether they are part of syntax as such or, as Chomsky suggests, part of the interface between syntax and the Conceptual-Intentional interface. If this is on the right track, this peculiar fact about language may turn out to be of a kind with many familiar physical phenomena, no more than "business as usual" in nature. If so, that would represent an advance in the understanding of language and a vindication of Chomsky's (2005) suspicion that "third factor" considerations play a significant role in determining the character of linguistic cognition.

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Appendix: Partial Catalog of Phrasal Patterns

In this appendix, we provide a partial list of possible phrasal patterns. This list is necessarily incomplete, as there is no *a priori* limit on the number of non-terminals that could be used to define the local "molecule" of structure; moreover, the number of systems of each degree increases dramatically as the degree increases. We provide tree diagrams and matrices for all orientations of all viable systems (prime and composite) of degree 1 and 2. For degree 3 systems, we provide only matrices, polynomials, and dominant eigenvalues, and only for the prime systems. For reasons of space, for the large class of degree 4 systems we omit both trees and matrices, providing only the characteristic polynomials and dominant eigenvalues of the prime systems.

   Let us recall that the systems of interest are binary-branching, discrete infinite recursive patterns. They each contain a single kind of terminal, and n distinct non-terminals, with a designated "root" non-terminal. The root must dominate (not necessarily immediately) each of the distinct types of nodes (terminal and non-terminals). In the prime systems, each non-terminal must also dominate (again, not necessarily immediately dominate) the root; composite systems are precisely those in which at least one non-terminal does not meet this condition.

   These systems were found with the aid of a computer-assisted search. The characteristic polynomials and eigenvalues given here were calculated using the following site: <http://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert.htm>

A.1 Degree 1

There is only a single discrete infinite system in this class, what we call the Spine. Recall that in our tree diagrams, a triangle indicates another root-type non-terminal (recursion of other non-terminals is indicated with arrows), and black circles are terminals. As throughout, linear order is unimportant.

Matrix: Tree: Polynomial: Dominant eigenvalue:

[1] $x - 1$ 1

Figure 8. Degree 1 system.

Intuitively, we can think of this system as corresponding to the following phrase structure rule: $XP \rightarrow X^0\ XP$. Note that the other naïve possibilities ($XP \rightarrow XP\ XP$, $XP \rightarrow X^0\ X^0$), what we call the Bush and the Pair, respectively, are either not discrete (the Bush has no terminals) or not infinite (the Pair doesn't grow).

A.2 Degree 2

We give the prime systems first. There are two distinct systems, with two "orientations" each (the different orientations result from distinct choices of which non-terminal is designated as the root). The different orientations have distinct matrices and tree forms, but identical polynomials and eigenvalues.

Matrix:	Tree:	Polynomial:	Dominant eigenvalue:
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$		$x^2 - x - 1$	1.6180...
$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$			
$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$		$x^2 - 2$	1.4142...
$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$			
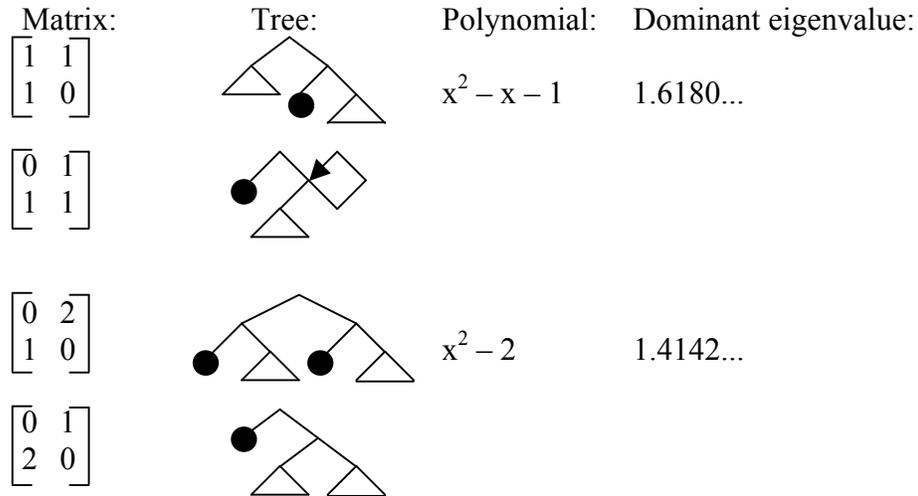
Figure 9. Prime systems of degree 2.

There are two composite systems in this class as well, what we call the Spine of Spines and the Spine of Pairs.

Matrix:	Tree:	Polynomial:	Dominant eigenvalue:
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$		$x^2 - 2x + 1$	1
$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$		$x^2 - x$	1
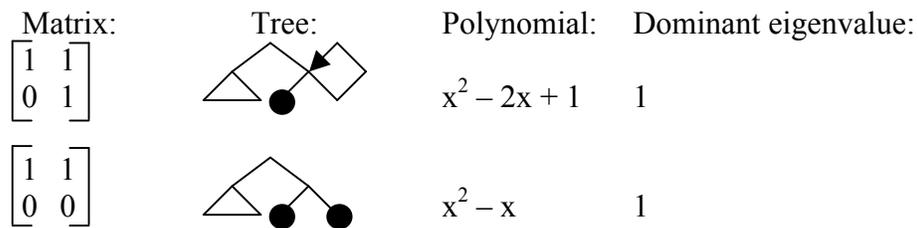
Figure 10. Composite systems of degree 2.

A.3 Degree 3

We omit trees for all systems, and all information about composite systems in this class. The prime systems of degree 3 admit three rootward orientations, so the list below gives the three matrices corresponding to each orientation in a single row, together with their shared polynomial and dominant eigenvalue.

Matrices:			Polynomial:	Dominant eigenvalue:
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	$x^3 - x^2 - x - 1$	1.8383...

$$\begin{bmatrix} 0 & 0 & \overline{1} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{1} \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \overline{0} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ $x^3 - x^2 - 2x + 1$ 1.8019...

$$\begin{bmatrix} 0 & 1 & \overline{1} \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & \overline{0} \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - 2x - 2$ 1.7693...

$$\begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \overline{0} \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \overline{0} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - 2x^2 + x - 1$ 1.7548...

$$\begin{bmatrix} 0 & 1 & \overline{1} \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{0} \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & \overline{0} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ $x^3 - 3x$ 1.7321...

$$\begin{bmatrix} 1 & 1 & \overline{0} \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & \overline{0} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - x^2 - 2$ 1.6956...

$$\begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & \overline{0} \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \overline{0} \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - x^2 - 2$ 1.6956...[14]

$$\begin{bmatrix} 0 & 2 & \overline{0} \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & \overline{0} \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \overline{1} \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$ $x^3 - 4$ 1.5874...

$$\begin{bmatrix} 0 & 1 & \overline{0} \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{1} \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \overline{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ $x^3 - x - 2$ 1.5214...

$$\begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \overline{0} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - x^2 - 1$ 1.4656...

$$\begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{0} \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{1} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - x - 1$ 1.3247...

$$\begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \overline{0} \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & \overline{0} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ $x^3 - 2$ 1.2599...

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[14] This is not an error; this system and the one immediately above it are genuinely distinct phrasal patterns, but happen to have identical polynomials and dominant eigenvalues.

A.4 Degree 4

Finally, we list the prime systems of degree 4. They are numerous enough that, to save space, we present them in a table, indicating only their polynomials and dominant eigenvalues, omitting the matrices and tree diagrams.

All the prime systems in this class have four orientations, reflecting distinct choices of non-terminal as root, with all four sharing the same polynomial and dominant eigenvalue. However, as we saw with the value 1.6956... in the degree 3 systems, some genuinely distinct patterns happen to share the same polynomial and eigenvalue. We also find one dominant eigenvalue (the one associated with the nonagon) for distinct systems with distinct polynomials.

Eigenvalue (growth factor)	Polynomial(s)	Notes
1.92756	$x^4-x^3-x^2-x-1$	Endocentric, Pisot
1.92129	$x^4-x^3-x^2-2x+1$	
1.91439	$x^4-x^3-2x^2+x-1$	
1.90517	$x^4-x^3-2x^2+1$	Pisot
1.89932	x^4-2x^2-2x-2	
1.89718	$x^4-2x^3+x^2-x-1$	
1.89329	x^4-2x^2-3x	
1.88721	x^4-3x^2-2	
1.88320	$x^4-2x^3+x^2-2x+1$	
1.87939	$x^4-x^3-3x^2+2x+1$ & x^4-3x^2-x	Nonagon, 2 polynomials.
1.87371	x^4-x^3-2x-2	
1.87018	x^4-3x^2-2x+2	
1.86676	x^4-2x^3+x-1	Pisot
1.86371	x^4-x^3-3x	
1.85356	$x^4-x^3-x^2-2$	
1.85163	x^4-x^3-4x+2	
1.84776	x^4-4x^2+2	Octagon
1.83509	x^4-4x-4	
1.82462	$x^4-x^3-2x^2+2x-2$	
1.82105	$x^4-x^3-x^2-2x+2$	
1.81917	$x^4-3x^3+3x^2-x-1$	
1.81712	x^4-6x	
1.80843	x^4-x^2-3x-2	
1.79891	x^4-2x^2-4	
1.79632	x^4-x^2-4x	
1.79431	x^4-x^3-2x-1	
1.79004	$x^4-2x^3+x^2-2$	
1.78537	x^4-2x^2-x-2	
1.74840	x^4-x^3-4	
1.74553	x^4-x^2-3x-1	
1.72775	x^4-4x-2	
1.72534	x^4-x^3-x-2	
1.72208	$x^4-x^3-x^2-x+1$	
1.71667	x^4-2x^3+2x-2	

1.71064	x^4-2x^2-x-1	
1.70211	x^4-3x^2-x+2	
1.69028	$x^4-x^3-2x^2+2x-1$	
1.68377	x^4-2x^2-2x+1	
1.68179	x^4-8	
1.67170	x^4-x^2-3x	
1.66980	$x^4-x^3-x^2+x-2$	
1.65440	x^4-x^3-3x+2	
1.65289	x^4-2x^2-2	
1.64293	x^4-2x-4	
1.60049	x^4-x^2-4	
1.56638	x^4-x^2-x-2	
1.55898	x^4-x^3-2x+1	
1.55377	x^4-2x^2-1	
1.54369	x^4-x^3-2	
1.51288	$x^4-x^3-x^2+x-1$	
1.49453	x^4-2x-2	
1.49022	x^4-2x^2-x+1	
1.44225	x^4-3x	
1.39534	x^4-2x-1	
1.38028	x^4-x^3-1	Pisot
1.35321	x^4-x-2	
1.27202	x^4-x^2-1	
1.22074	x^4-x-1	
1.18921	x^4-2	